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On the Maximum Transform

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Let the real-valued functions $f(x)$, $g(x)$ of $x = (x_1, x_2, \cdots, x_n)$ be defined and continuous on $x_i \geq 0$, $i = 1, 2, \cdots, n$. The *maximum convolution* $f \oplus g$ is given by $(f \oplus g)(x) = \max_{u+v=x} [f(u) + g(v)]$. In this paper we characterize the continuous transformations T with the property (*) $T(f \oplus g) = Tf + Tg$. More precisely, let f be *admissible* in case the projection of its graph on a line with direction $(-\xi, 1)$ is bounded in this direction for each $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ with $\xi_i > 0$, $i = 1, 2, \cdots, n$. Let the *maximum transform* $\varphi = Mf$ be defined for such ξ by $\varphi(\xi) = -\sup_{x_i \geq 0} [-(\xi, x) + f(x)]$, $(\xi, x) = \Sigma_i \xi_i x_i$. Let $\mathscr{A}$ denote the class of admissible functions, and $\mathscr{B}$ denote the class of images of members of $\mathscr{A}$ under M. (The mapping M is also defined from $\mathscr{B}$ to $\mathscr{A}$ by $(M\varphi)(x) = -\sup_{\xi_i > 0} [-(x, \xi) + \varphi(\xi)]$.) Call f in $\mathscr{A}$ the "limit" of f_n in $\mathscr{A}$ in case f_n, $\varphi_n = Mf_n$ converge pointwise to f, $\varphi = Mf$, respectively; similarly for ψ being the limit of ψ_n in $\mathscr{B}$. Let T map each $f \in \mathscr{A}$ into an element $Tf \in \mathscr{T}$, where $\mathscr{T}$ is an appropriately general space. The principal result is the following: T is continuous and has property (*) if and only if $T = \lambda M$ where λ is a continuous transformation from $\mathscr{B}$ to $\mathscr{T}$ such that $\lambda(a\varphi_1 + b\varphi_2) = a\lambda\varphi_1 + b\lambda\varphi_2$, for arbitrary φ_1, φ_2 in $\mathscr{B}$, $a \geq 0$, $b \geq 0$.

INTRODUCTION

In various problems of optimal allocation of resources there arises what we have chosen to call (see [1]) the *maximum convolution* $f \oplus g$ of two functions f and g; this is defined by

$$(f \oplus g)(x) = \max_{u+v=x} [f(u) + g(v)]; \tag{1}$$

here $x = (x_1, x_2, \cdots, x_n)$, u and v denote points in Euclidean n-space restricted to the set $A = \{x \mid x_i \geq 0, \ i = 1, 2, \cdots, n\}$. We have studied (nonlinear) transformations T having the property

$$T(f \oplus g) = Tf + Tg. \tag{2}$$

In the present paper we continue this investigation by defining a notion of "limit" in a space $\mathscr{A}$ of admissible functions to which the operation (1) applies, and studying arbitrary continuous transformations T from $\mathscr{A}$ to a general space $\mathscr{T}$ which satisfy (2). The main result (see Theorem 2) is that such a transformation is characterized as a composition of two transformations

$$T = \lambda M, \tag{3}$$

where M is what we have called the ***maximum transform***, and λ is any continuous transformations from the space $\mathscr{B} = M\mathscr{A}$ of maximum transforms to $\mathscr{T}$ having the linearity property

$$\lambda(a_1\varphi_1 + a_2\varphi_2) = a_1\lambda\varphi_1 + a_2\lambda\varphi_2, \tag{4}$$

φ_1, φ_2 in $\mathscr{B}$ and a_1, a_2 nonnegative real numbers. The maximum transform $\varphi = Mf$ of an admissible function f is defined by

$$\varphi(\xi) = - \sup_{x \in A} [- (\xi, x) + f(x)] \tag{5}$$

where we confine $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ to the set $B^0 = \{\xi \mid \xi_i > 0, \ i = 1, 2, \cdots, n\}$; (ξ, x) denotes $\sum_{i=1}^{n} \xi_i x_i$. That the transformation M itself has property (2) may be verified by a direct argument (see [1]). The transformation M has been studied by Fenchel for general concave functions f (see [2]); he terms Mf the "conjugate" of f.

The discussion that follows will be based upon the paper [1]; for completeness, necessary definitions will be repeated here.

Admissible Functions

Let f be a real-valued function on A and let $\xi \in B^0$; then the half-space

$$Z - (\xi, X) \leq b$$

is the *support* of f in the direction $(-\xi, 1)$ in case

$$f(x) - (\xi, x) \leq b, \qquad x \in A,$$

and b (finite) is the least number for which this holds. The function f is called *admissible* in case it is continuous on A and has a support for each direction $(-\xi, 1)$, $\xi \in B^0$. Let

$$\mathscr{A} = \text{class of admissible functions.}$$

The maximum convolution $\oplus$ is closed, associative, and commutative, in the class $\mathscr{A}$.

Associated with an admissible f is its maximum transform

$$\varphi = Mf$$

given by (5) and taken to be defined on B^0; geometrically, $-\varphi(\xi)$ is the z-intercept of the hyperplane of the support of f in the direction $(-\xi, 1)$. Let

$$\mathscr{B} = \text{class of functions } Mf, \qquad f \in \mathscr{A}.$$

If $\varphi \in \mathscr{B}$, then φ is concave, increasing ($\xi \geq \eta$ implies $\varphi(\xi) \geq \varphi(\eta)$), and bounded from above on B^0; in fact, $\mathscr{B}$ can be characterized as this class of concave functions. The transformation M also acts on $\mathscr{B}$ according to the analog of (5),

$$(M\varphi)(x) = -\sup_{\xi \in B^0} [-(x, \xi) + \varphi(\xi)], \qquad x \in A; \tag{6}$$

the resulting function $M\varphi$ is concave and belongs to $\mathscr{A}$.

The mapping M from $\mathscr{B}$ to $\mathscr{A}$ can be made more explicit as follows. For an admissible f, we define the admissible function

$$f^+(x) = \max_{0 \leq y \leq x} f(y)$$

("$\leq$" means inequality for every component). In general, let $[g]$ denote the set of points (x, z) in $(n + 1)$-space given by

$$[g] = \{(x, z) \mid z \leq g(x), x \in A\}.$$

Now consider the closure H of the convex hull of $[f^+]$ (this is also the intersection of all supports of f with directions $(-\xi, 1)$, $\xi \in B^0$). Let $\bar{f}$ be the function which describes the "cap" of H, i.e.,

$$\bar{f}(x) = \max_{(x,z) \in H} z.$$

Then $\bar{f}$ is an admissible concave function, and

$$M\varphi = \bar{f}, \qquad \text{where} \qquad \varphi = Mf. \tag{7}$$

Also $Mf = Mg$ if and only if $\bar{f} = \bar{g}$, with f, g in $\mathscr{A}$. The functions $\bar{f}$ comprise the subclass $\mathscr{A}^*$ of admissible functions f which are concave and increasing. The maximum transform is one-to-one between $\mathscr{A}^*$ and $\mathscr{B}$, and $MMf = f$, $MM\varphi = \varphi$ for any f in $\mathscr{A}^*$ and any φ in $\mathscr{B}$.

The classes $\mathscr{A}^*$ and $\mathscr{B}$ are closed under the operation of forming the infimum, in the following sense: if $\{f_\alpha\}$ is a family of functions of $\mathscr{A}^*$ for which

$$f(x) = \inf_\alpha f_\alpha(x)$$

is finite for each $x \in A$, then f is also in $\mathscr{A}^*$. Similarly, if

$$\psi(\xi) = \inf_\alpha \psi_\alpha(\xi)$$

is finite for each $\xi \in B^0$, then ψ belongs to $\mathscr{B}$ when the ψ_α do. The classes $\mathscr{A}$, $\mathscr{A}^*$ and $\mathscr{B}$ are each closed under addition, and multiplication by non-negative scalars. In particular, if $\varphi_1, \varphi_2 \in \mathscr{B}$ and $g_1 = M\varphi_1$, $g_2 = M\varphi_2$, then $\varphi_1 + \varphi_2 = M(g_1 \oplus g_2)$, $M(\varphi_1 + \varphi_2) = g_1 \oplus g_2$; also, for $b \geq 0$, $b\varphi_1 = Mh$ where $h(x) = bg_1(x/b)$ or 0, according as $b > 0$ or $b = 0$.

Limits

We introduce the following definition of "limit" in the spaces $\mathscr{A}$ and $\mathscr{B}$.

Definition. *Let f and f_n, $n = 1, 2, 3, \cdots$ belong to $\mathscr{A}$; then f_n is said to have f as a limit in case*

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} \varphi_n(\xi) = \varphi(\xi), \qquad \textit{for} \qquad x \in A \quad \textit{and} \quad \xi \in B^0,$$

where $\varphi_n = Mf_n$, $\varphi = Mf$; in this case, we write $f_n \to f$. Similarly, we write $\psi_n \to x$ with ψ, ψ_n in $\mathscr{B}$ in case for each $\xi \in B^0$ and $x \in A$, $\psi_n(\xi)$ and $(M\psi_n)(x)$ converge to $\psi(\xi)$ and $(M\psi)(x)$, respectively.

Lemma 1. *If $f_n \to f$ in $\mathscr{A}$, then there exist F_1, F_2 in $\mathscr{A}^*$ such that*

$$F_1 \leq \bar{f}_n \leq F_2, \qquad MF_2 \leq Mf_n = M\bar{f}_n \leq MF_1.$$

Similarly, if $\psi_n \to x$ in $\mathscr{B}$, then there exist Ψ_1, Ψ_2 in $\mathscr{B}$ such that

$$\Psi_1 \leq \psi_n \leq \Psi_2, \qquad M\Psi_2 \leq M\psi_n \leq M\Psi_1.$$

PROOF. By hypothesis, $f_n(x)$, $\varphi_n(\xi)$ converge to $f(x)$, $\varphi(\xi)$ respectively, when $\varphi_n = Mf_n = M\bar{f}_n$, $\varphi = Mf = M\bar{f}$. It follows that

$$F_1(x) = \inf_n \bar{f}_n(x), \qquad \Phi_2(\xi) = \inf_n \varphi_n(\xi)$$

are finite for each $x \in A$ and $\xi \in B^0$; thus, $F_1 \in \mathscr{A}^*$, $\Phi_2 \in \mathscr{B}$ with

$$F_1 \leq \bar{f}_n, \qquad \Phi_2 \leq \varphi_n.$$

From the definitions (5) and (6).

$$\varphi_n(\xi) = (M\bar{f}_n)(\xi) \leq (MF_1)(\xi),$$
$$\bar{f}_n(x) = (M\varphi_n)(x) \leq (M\Phi_2)(x).$$

Letting $F_2 = M\Phi_2$, we obtain the desired inequalities. The proof for $\psi_n \to \psi$ is similar.

LEMMA 2. *If $f_n \to f$ in $\mathscr{A}$, then $\bar{f}_n \to \bar{f}$ in $\mathscr{A}$.*

PROOF. By assumption, $f_n(x)$ tends to $f(x)$ and $\varphi_n(\xi)$ to $\varphi(\xi)$, where $\varphi_n = Mf_n = M\bar{f}_n$ and $\varphi = Mf = M\bar{f}$. By Lemma 1, $\bar{f}_n$ is uniformly bounded on every compact subset of A. Now let $\{\bar{f}_j\}$ be an arbitrary subsequence of $\{\bar{f}_n\}$; there exists a further subsequence, say $\{\bar{f}_k\}$, converging to some concave function g for each $x \in A$ (in fact, the convergence is uniform on every compact subset of A; see [2, p. 75]). We wish to show that $g = \bar{f}$. Let

$$g_k(x) = \inf_{k' \geq k} \bar{f}_{k'}(x).$$

Then $g_k \leq g_{k+1}$ and $g_k(x)$ converges to $g(x)$. Since $f_k(x)$ converges to $f(x)$,

$$\varliminf_{k \to \infty} f_k^+ \geq f^+.$$

This, together with $\bar{f}_k \geq f_k^+$, implies $\varliminf \bar{f}_k \geq f^+$, or $g \geq f^+$. Since $g \in \mathscr{A}^*$, the last inequality gives $g \geq \bar{f}$.

For the reverse inequality, let ξ be an arbitrary point on B^0. Let $\pi(f)$ denote the support of f with direction $(-\xi, 1)$. Since $\varphi_k = M\bar{f}_k$ and $\varphi = M\bar{f}$, the equality $\lim_k \varphi_k(\xi) = \varphi(\xi)$ implies that $\pi(\bar{f}_k)$ tends to $\pi(\bar{f})$. For $k' \geq k$, $\pi(g_k)$ is not above $\pi(\bar{f}_{k'})$; it follows that $\pi(g_k)$ is not above $\pi(\bar{f})$. Consequently, $\pi(g)$ is not above $\pi(\bar{f})$, by the pointwise convergence (everywhere) of $g_k(x)$ to $g(x)$. From the fact that $\bar{f}$ is the "cap" of the intersection of all its supports with directions $(-\xi, 1)$, $\xi \in B^0$, it follows that $\bar{f} \geq g$. Thus, $\bar{f} = g$.

Since $\{f_j\}$ was an arbitrary subsequence of the original sequence $\{f_n\}$, this proves that $f_n(x)$ converges to $f(x)$. By assumption, $(Mf_n)(\xi)$ converges to $(Mf)(\xi)$. Hence $f_n \to f$, as desired.

LEMMA 3. *If* $\psi_n \to \psi$ *in* $\mathscr{B}$, *then* $\lim \psi_n(\xi) = \psi(\xi)$ *uniformly on compact subsets of* B^0 *and* $\lim (M\psi_n)(x) = (M\psi)(x)$ *uniformly on compact subsets of* A.

PROOF. The proof is essentially that of Lemma 2. By Lemma 1, an arbitrary subsequence of ψ_n has itself a subsequence converging to a function φ in $\mathscr{B}$, uniformly on compact subsets of B^0. Following the argument of Lemma 2, we obtain $\psi = \varphi$; from this, the required convergence of the lemma follows. A similar argument applies to $M\psi_n$ in $\mathscr{A}$.

CONTINUOUS TRANSFORMATIONS

Our interest is in transformations T which map each $f \in \mathscr{A}$ into an element Tf of a space $\mathscr{T}$. We take $\mathscr{T}$ to be a topological space of elements $\alpha, \beta, \cdots$ subject to a commutative binary operation $\alpha + \beta$ and real scalar multiplication $a\alpha$. The space $\mathscr{T}$ is assumed to be a linear space except that $(\mathscr{T}, +)$ is only required to form a semigroup (with cancellation), and scalar multiplication is restricted to nonnegative real numbers. Such a transformation is understood to be *continuous* in case

$$f_n \to f \text{ in } \mathscr{A} \text{ implies } Tf_n \text{ has } Tf \text{ as limit in } \mathscr{T};$$

similarly for transformations from $\mathscr{B}$ to $\mathscr{T}$.

THEOREM 1. *The space* $\mathscr{B}$ *is of the type* $\mathscr{T}$ *described above. The transformation* M *from* $\mathscr{A}$ *to* $\mathscr{B}$ *is continuous and has the property* (2).

PROOF. The topology in $\mathscr{B}$ is that determined by the earlier definition $\psi_n \to \psi$ (using Lemma 3). Addition and nonnegative scalar multiplication are the ordinary ones, and closure with respect to these operations was pointed out earlier. To show that $\varphi + \psi$ is continuous in (φ, ψ) suppose $\varphi_m \to \varphi$, $\psi_n \to \psi$, with images f_m, f, g_n, g in $\mathscr{A}^*$. It is necessary to show that $\varphi_m + \psi_n$, $M(\varphi_m + \psi_n)$ converge pointwise to $\varphi = \psi$, $M(\varphi + \psi)$, respectively, as m, n tend to ∞ independently. The first convergence is immediate; for the second, we have $\varphi_m + \psi_n = M(f_m \oplus g_n)$, or

$$M(\varphi_m + \psi_n) = f_m \oplus g_n.$$

By Lemma 3, $f_m(x), g_n(x)$ converge uniformly to $f(x), g(x)$ on compact subsets; hence the right side tends to $f \oplus g$, or $M(\varphi + \psi)$, for each $x \in A$, as desired.

To show that $a\psi$ is continuous in (a, ψ) let a_m tend to a and $\psi_n \to \psi$. We have $a_m\psi_n(\xi)$ approaching $a\psi(\xi)$ for each $\xi \in B^0$. Now

$$[M(a_m\psi_n)](x) = \begin{cases} a_m g_n(x/a_m) & \text{for} \quad a_m > 0, \\ 0 & \text{for} \quad a_m = 0. \end{cases} \tag{8}$$

Suppose $a > 0$. In this case, we need only consider the first equation in (8); from the uniform convergence of g_n on compact subsets we obtain $ag(x/a)$ as the limit on the right of (8) for each $x \in A$; this equals $M(a\psi)$, as desired. Now suppose $a = 0$. We wish to show the limit is $M(0\psi) = 0$ in this case; thus, we may again restrict attention to the first line of (8). From Lemma 1, $\Psi \leq \psi_n \leq K$, with $\Psi \in \mathscr{B}$ and K a constant; let $G = M\Psi$. Multiply the last inequality through by a_m and operate with M. Since $M(a_mK) = -a_mK$ and $(\xi, x) \geq \Psi(\xi) + G(x)$, we obtain

$$\begin{aligned} -a_mK \leq a_m g_n(x/a_m) &\leq a_m G(x/a_m) \\ &\leq (\xi, x) - a_m\Psi(\xi); \end{aligned}$$

these inequalities hold for arbitrary $x \in A$, $\xi \in B^0$. It follows that the limit in (8) is 0, as desired.

The above argument establishes the first statement of the theorem. The continuity of M follows from Lemma 2 and the identity $MMg = \bar{g}$ for $g \in \mathscr{A}$; property (2) was noted earlier. This completes the proof of the theorem.

Our principal result is expressed in the following theorem, which states that an arbitrary T satisfying (2) consists of M followed by a linear transformation.

THEOREM 2. *Let T be a transformation which maps each $f \in \mathscr{A}$ into some element Tf of a space $\mathscr{T}$ of the type described above. Then T is continuous and has property (2) if and only if*

$$T = \lambda M \tag{9}$$

where λ is a continuous transformation from $\mathscr{B}$ to $\mathscr{T}$ such that

$$\lambda(a\varphi + b\psi) = a\lambda\varphi + b\lambda\psi \tag{10}$$

whenever $a \geq 0$, $b \geq 0$, and $\varphi, \psi \in \mathscr{B}$.

PROOF. The key to the proof is the following identity for any $f \in \mathscr{A}$:

$$f \oplus f \oplus \cdots \oplus f = f \oplus f \oplus \cdots \oplus f,$$

where each side has $n + 1$ terms. This is established in [1]. Now suppose T is continuous and has property (2); applying T to both sides yields

$$T\bar{f} = Tf.$$

From this and $MMf = \bar{f}$,

$$T = \lambda M, \qquad \text{where} \qquad \lambda = TM.$$

The transformation λ operates from $\mathscr{B}$ to $\mathscr{T}$; the continuity of T from $\mathscr{A}$ to $\mathscr{T}$ and that of M from $\mathscr{B}$ to $\mathscr{A}$ (a consequence of the definition of $\varphi_n \to \varphi$), implies the continuity of λ. To establish (10), let $\bar{f} = M\varphi$, $\bar{g} = M\psi$; then $\lambda(\varphi + \psi) = \lambda M(\bar{f} \oplus \bar{g}) = TMM(\bar{f} \oplus \bar{g}) = T(\bar{f} \oplus \bar{g}) = Tf + Tg = \lambda\varphi + \lambda\psi$, where we have used the fact that $\bar{f} \oplus \bar{g}$ belongs to $\mathscr{A}^*$ when $\bar{f}, \bar{g}$ do.

This establishes (10) for $a = b = 1$. From this follows

$$\lambda(a_m\varphi) = a_m\lambda\varphi \tag{11}$$

for positive rationals a_m. Let $a \geq 0$ be arbitrary, and let $a_m > 0$ converge to a. By Theorem 1, $a_m\varphi \to a\varphi$ in $\mathscr{B}$ and by the assumptions on $\mathscr{T}$, $a_m\lambda\varphi$ has $a\lambda\varphi$ as limit. Taking the limit in (11) and using the continuity of λ gives (10) with $b = 0$. It now follows that (10) holds as stated, completing the first half of the proof. The converse is an immediate consequence of Theorem 1.

References

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